Design Criteria for Axially Loaded Cylindrical Shells

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A method of elastic stability analysis for axially compressed cylindrical shells is presented, and applied to all types of cylindrical shells for which experimental results are available. The method is clearly conservative because it predicts critical loads below the test results for every one of the more than 250 test specimens included in the evaluation. A review of previous efforts to establish design criteria is included and it appears that the method recommended here is superior to presently employed methods. Still it must be considered an interim solution; continued research will eventually lead to more satisfactory design principles.

Nomenclature

A_{ij} = see Eqs. (A13)
a,b,c = see Eqs. (A11)
B_i = see Eqs. (A15)
C = see Eq. (A16)
C_{ij} = see Eq. (A1)
$\bar{C}_{44},\bar{C}_{55}$ = see Eq. (A6)
$\bar{C}_{11},\bar{C}_{14}$ = see Eq. (A10)
E_c = core modulus
K_i = see Eqs. (A12) and (A14)
L = length of cylinder
M_1,M_2,M_T = bending and twisting moments
m = number of waves in axial direction
N_1,N_2,N_{12} = membrane forces
N_{WC} = wide column load
N_{CR} = critical line load for design purposes
N_{CL} = critical line load according to classical theory
n = number of waves in circumferential direction
P_{DES} = design load for a cylinder
$P_{CLASSICAL}$ = critical load on cylinder according to classical theories
P_{50},P_{90},P_{99} = critical load for cylinder for 50%, 90%, and 99% probability
p = external pressure
$\bar{p}$ = $(p/E)(R/t)^2$
p_c = interface pressure (cylinder to core)
R = shell radius
$(R/t)_e$ = effective radius to thickness ratio
t = thickness of monocoque shell
$\bar{t}$ = distance with respect to which imperfections are normalized; t for the monocoque shell
u_{mn} = see Eq. (A3)
v_{mn} = see Eq. (A4)
α,β,γ = see Eq. (A17)
$\epsilon_1,\epsilon_2,\epsilon_{12}$ = middle surface strains
$\kappa_1,\kappa_2,\kappa_{12}$ = changes of curvature
λ = see Eq. (A10)
$\bar{\lambda}$ = see Eq. (A2)
μ = amplitude of initial imperfection (normalized with respect to $\bar{t}$)
ν_c = Poisson's ratio for core
$\bar{\sigma}_{CR}$ = critical stress normalized with respect to the quantity $E(t/R)$
ϕ = reduction factor
ψ_{mn} = core modulus
ψ_0,ψ_1 = core modulii, for prebuckling and incremental displacements, respectively

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Introduction

THE theory for buckling of thin shells fails in many cases to yield results in agreement with experiments. Cylindrical shells under axial compression show particularly poor agreement. For very thin monocoque cylinders, loads have been obtained at experiments which are as low as about one fifth of the theoretical values.

In Ref. 1 it is shown that the experimental values of Ref. 2 for stiffened shells range between 57 and 79% of the theoretical values. These values are well in line with those obtained for monocoque shells of similar "effective" thickness. Another example of stiffened shells that give test results well below the prediction of theory are the large and very carefully manufactured cylinders discussed in Ref. 3. It seems that the allegation, sometimes made in the literature, that the classical buckling theory is applicable for stiffened shells, is repudiated by experiments. Methods to predict reduced buckling loads for such shells are needed as well as more adequate methods for monocoque cylinder analysis.

Recent research efforts in the field of thin shell buckling have led to a good understanding of the basic reasons for the poor agreement between test and theory. However, the results of the research have not been utilized to devise better methods of practical analysis. Therefore, it is attempted here to establish design procedures which are based on the knowledge that has been acquired over the years.

The first analysis of the postbuckling behavior of axially loaded cylinders[4] indicated that the minimum postbuckling load was about one third of the classical buckling load and thus reasonably close to the average of the available test results. It was suggested therefore that the minimum postbuckling load be used as a design limit and it was somewhat unfortunately termed the lower buckling load. A more accurate postbuckling analysis[5] indicates that the minimum postbuckling load is not suitable as a design load. It was never widely accepted as such. Instead, the classical buckling load was used together with a reduction factor which generally was chosen to be about 0.25.

In a paper by Donnell and Wan in 1950,[6] geometrical imperfections were recognized as the major reason for the discrepancy between test and theory. Such imperfections were included in their analysis. However, for a tractable analysis, certain simplifications had to be introduced which reduced the analysis to a qualitative demonstration of the importance of imperfections.

As a consequence of the lack of an adequate theoretical analysis, the designers of cylindrical shells were forced to use empirical methods. In 1957[7] a first attempt was made to devise a design limit by use of a statistical analysis of available test results. For different probability levels, a reduction factor φ is given as a function of the radius to thickness ratio. Similar analyses were presented later in which some test results were added and others for one reason or another were excluded. Here the design curves of Ref. 8 are shown in Fig. 1. The disadvantage with the statistical analysis procedure of course is that some test results would affect the design although due to the manner of fabrication or to their size they are irrelevant.

For analyses such as that of Ref. 7, it appears that the number of available tests would constitute a sufficient statistical background as long as the buckling coefficient for a fixed probability is a function of only one variable, R/t. This is the case for longer shells but for shorter shells the shell length becomes an additional parameter and for this case the number of available tests is not quite satisfactory.

For orthotropic shells or stiffened shells, the number of influential parameters becomes so large that a purely empirical approach is out of the question. More or less conservative design principles have therefore been applied. For stringer stiffened cylinders it was quite generally assumed that the effect of curvature is negligible and thus that the wide column load should be applicable as a design limit. This principle has been applied also in the analysis of buckling between rings (panel buckling) for cylinders stiffened with rings as well as stringers. An approximate method to determine the ring size, such that general instability is avoided, was given by Shanley.[9] The Shanley method, being purely empirical and based on very few test results, is not reliable and besides it is restricted in application because it cannot be used for a case in which the stringers are oversized.

In Ref. 10 it was recognized that use of the wide column load as a design limit for stringer stiffened cylinders was unduly conservative. It was suggested that a term be added to the wide column load which corresponds to the curvature effect. This term was obtained as the difference between the classical buckling load and the wide column load multiplied by a reduction factor. After definition of an effective radius to thickness ratio, this factor could be obtained from test data for monocoque shells (as shown in Fig. 1).

In Ref. 11 a set of design rules are given for stiffened cylinders. The same basic approach is used for the selection of a reduction factor as in Ref. 10, but the importance of stiffener eccentricity is recognized. The method works well for the cases in which the wide column load is relatively high (short cylinders, core-filled cylinders) but for pressurized cylinders and ring-stiffened cylnders more rational design criteria are needed.

Recommendations

It appears from the discussion above that for some types of cylindrical shells a practical method of analysis is available, which works reasonably well. The question, thus, is if other methods can be found which are more general or which would cover the cases for which adequate methods now are lacking. In particular it appears possible to take advantage of the results of recent research efforts.

Although residual stresses and variations in modulus or in shell thickness certainly do influence the critical load, it appears likely that the deviations from the true form of the shell midsurface is the major reason for the discrepancy between theory and test. Thus it appears feasible to apply the theory presented by Koiter[12] in the establishment of methods of practical analysis. Such application was to some extent discussed in Ref. 13. An equation which is valid for small initial imperfections was given for the critical load in terms of the classical buckling load, the amplitude of the imperfections, and the so-called imperfection sensitivity parameter. This equation is valid only for the case in which the imperfections are proportional to the buckling mode. In addition to the difficulties connected with the establishment of a representative value for the amplitude of the imperfections, other

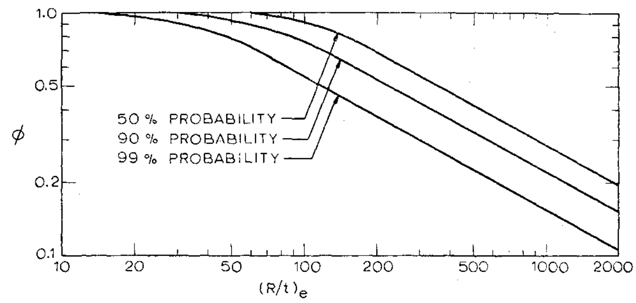

Fig. 1 Empirical factor φ for cylinders subjected to a uniform axial load.

problems arise. It is, for instance, pointed out in Ref. 13 that the lowest eigenvalue may correspond to a buckling mode which is not imperfection sensitive while a slightly higher eigenvalue corresponds to a sensitive mode shape. This situation certainly occurs for ring stiffened and for pressurized cylinders. Although it seems feasible to investigate a series of buckling modes, the choice of imperfection amplitude becomes in this case more difficult. In view of the fact that several buckling modes must be considered, such an analysis may well turn out to require more computer time than a rigorous solution of the nonlinear equations. It appears that more work is required before we can utilize the principles of Koiter's general theory in practical analysis.

A different approach was used here inasmuch as only axially symmetrical imperfections were included in the analysis. If it is assumed that the symmetrical component of the imperfections is of dominating influence, the following procedure can be followed. For a given shell, an equivalent monocoque cylinder is chosen on the basis of the effective radius to thickness ratio.[8] The reduction of the critical load for this infinitely long monocoque shell can then be found for instance from Fig. 1. The amplitude of the symmetrical imperfections which would result in such a reduction can then be established from the curve presented by Koiter.[14] It is then assumed that the cylinder under consideration has the same imperfection amplitude as the equivalent cylinder and thus the design load is found for the orthotropic, stiffened, short, core-filled, or pressurized shell.

The method outlined above was first applied in the cases for which adequate methods of practical analysis previously were available, that is for core-filled or short stringer-stiffened shells. For such cases the method with axially symmetric imperfections proved to be somewhat less satisfactory than the method based on the wide column load previously recommended in Ref. 10. As the wide column load is not affected by the addition of internal pressure or ring stiffening the "wide column load method" is definitely unacceptable for those cases. The "symmetric imperfection method" was applied to these cases and was found through comparison with test results to work reasonably well, although somewhat conservative. It was decided, therefore, that design principles would be recommended which were based on the assumption that both of these methods are conservative. Consequently, in any particular case, the higher of the predictions from the two methods would be used as a design limit. In Appendix A are given the final equations for the classical buckling load, for the lower bound (wide column load), and for the buckling load in the presence of axisymmetrical initial imperfections.

For orthotropic or stiffened cylinders the number of influential parameters is so large that it is impossible to present design curves which apply with any degree of generality. Therefore, a computer program is presented in Ref. 15 which gives design loads for axially loaded cylinders in accordance with the method recommended. In the following section results from this computer program are compared to available test results.

Results

In Refs. 16 and 17 results are presented for stiffened cylinders which are in such a range that agreement between test and theory may be expected. Such agreement seems to be at hand but the comparison is obscured by uncertainties about the edge conditions prevailing at test. These test results were consequently excluded from the comparison in Figs. 2–7. Also excluded were cases in which the buckling of the cylinder appeared to have been precipitated by local effects. Such cases are fiberglass cylinders with a radius to thickness ratio of 100 or less for which shear failure in the matrix appears to be the primary mode of failure and the stiffened cylinders for which the investigator had indicated that local buckling occurred substantially below the general instability load. Also excluded, of course, were the cases in which inelastic behavior substantially modifies the buckling load.

Comparisons between test and theory are shown for more than 250 cylinders of different types. In Fig. 2a, the test re-

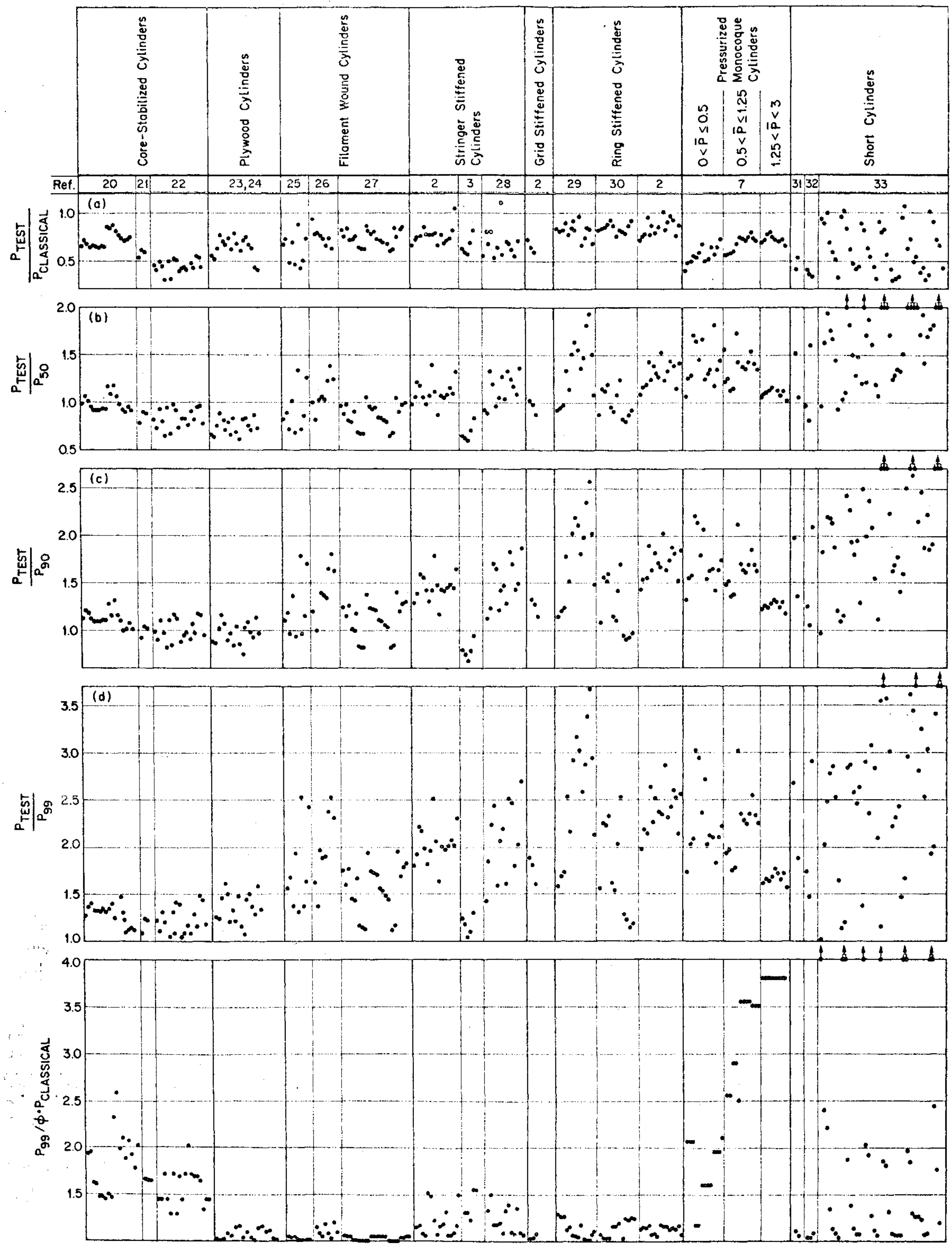

Fig. 2 Correlations between test results and a) classical theory, b) 50% probability predictions, c) 90% probability predictions, d) 99% probability predictions, and e) comparison of 99% probability predictions with predictions based on classical theory times φ for a monocoque shell with the same effective R/t ratio.

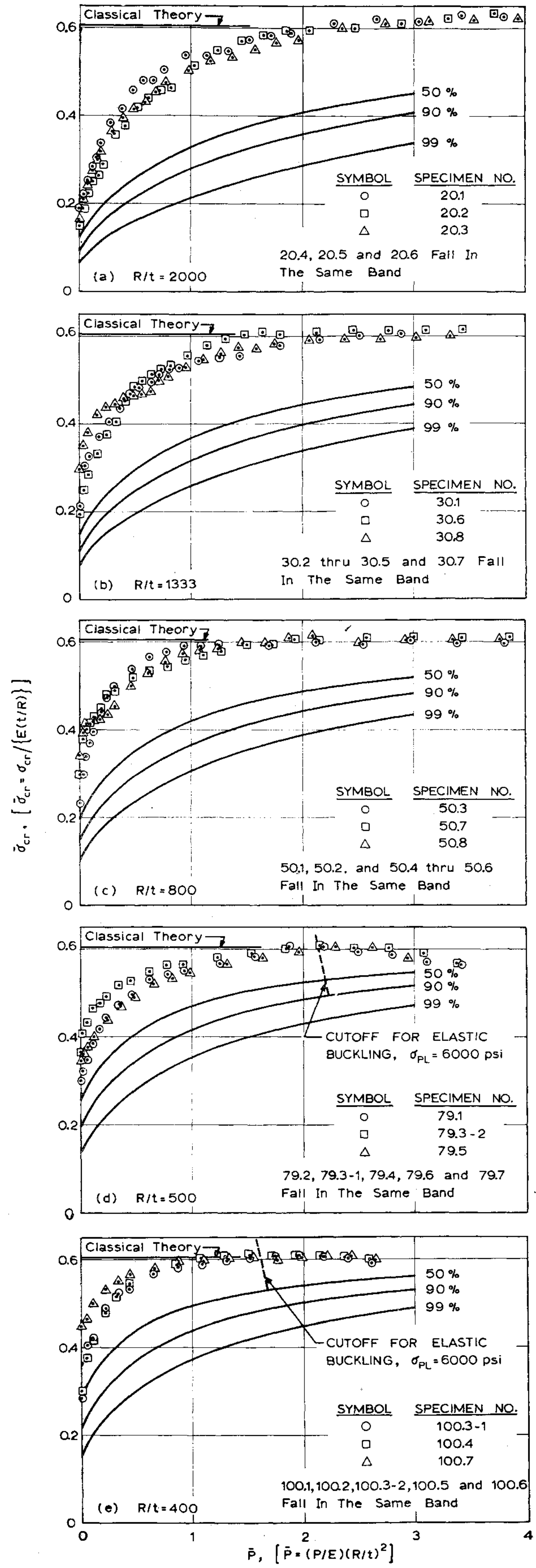

Fig. 3 Comparison between the pressurized cylinder data of Ref. 34 and predictions for three probability levels, and a) R/t values of 2000, b) 1333, c) 800, d) 500, and e) 400.

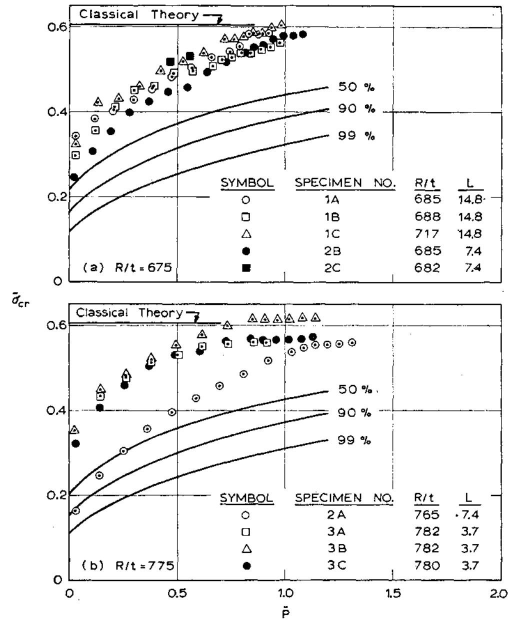

Fig. 4 Comparison between the pressurized cylinder data of Ref. 35 and predictions for three probability levels a) R/t = 675 and b) R/t = 775.

sults are compared to the critical load according to classical theory. While all the theoretical results are for cylinders with simply supported edges the test conditions are probably more likely to correspond to clamped edges. For most cases it is believed that the influence of the edge conditions is insignificant but there may be cases especially for stringer-stiffened shells in which the test results would have been considerably lower if the conditions of simply supported edges could have been realized. This is illustrated in the tables of Ref. 1 for some stringer stiffened and some filament wound cylinders, and it explains why three of the tests give results above the classical load.

In Figs. 2b–2d, the test results are compared to the predictions by the methods of analysis recommended here. Figure 3b corresponds to the use of the 50% probability curve for the monocoque cylinders (see Fig. 1), while Figs. 2c and 2d compare results of tests to predictions based on 90 and 99% probability, respectively. It appears from Fig. 2c that the results based on 90% probability would not be entirely "safe" and that the 99% probability curve thus should be

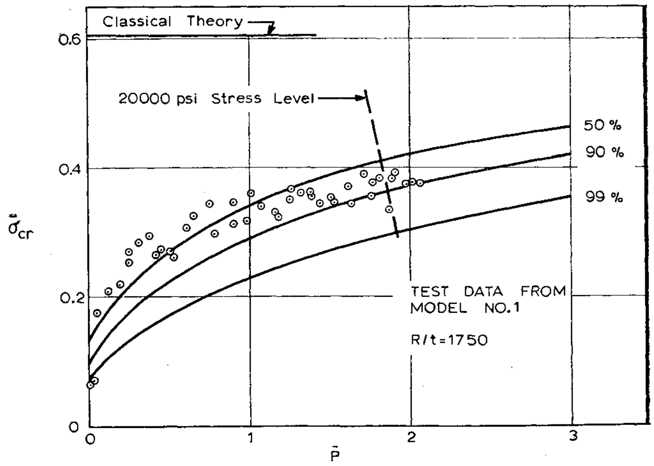

Fig. 5 Comparison between the pressurized cylinder data of Ref. 36 and predictions for three probability levels.

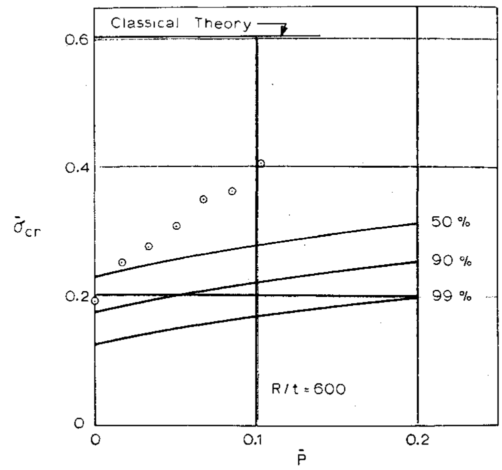

Fig. 6 Comparison between the pressurized cylinder data of Ref. 37 and predictions for three probability levels.

recommended for design. For pressurized cylinders, a series of test results at different values of the internal pressure were often obtained from the same test specimen. As the format of Fig. 2 is not suitable for such cases, comparisons between tests of this type and theory are presented in Figs. 3–6.

It is evident that the classical buckling load is not a suitable design limit for any of these classes of axially compressed cylinders. Although it sometimes has been stated in the literature that for one type of cylindrical shell or another the theory would be applicable, the designer is generally more prudent and applies conservative methods. As an example, for stringer-stiffened shells the wide column load is sometimes used as a design limit and in other cases a part of the "curvature effect" is added.[10] The present method is less conservative because the corresponding design limit is either equal to or higher than that of Ref. 10. For other cases it is a common procedure to apply simply the same reduction factor as for the infinite monocoque shell with the same effective radius to thickness ratio. In Fig. 2e the predictions of such a method ($P_{DES} = \varphi \cdot P_{CLASSICAL}$) are compared to those of the method recommended here (P_{99}). It is seen that the present method gives the same or higher values in all cases and that sometimes the difference is substantial.

It is clear, thus, that the design principles recommended here will lead to more economic design than the methods which generally are in use. At the same time they should be entirely safe as out of more than 250 test specimen of many different types every one failed at a value above the recommended design limit. It seems that improvements may be possible through minor modifications of the method. The choice of the curves in Fig. 1 and the definition of an effective radius to the thickness ratio may for instance be questioned. Although it is felt that the method recommended here represents a clear advantage over present design practices, it is still an interim solution acceptable only because totally satisfactory methods are not available.

To a designer who is inclined to disagree with the design principles recommended here, the collection of test results and the comparison to the classical buckling load should still be of interest.

Appendix A: Final Equations

The aforementioned method of practical analysis is based on the equations which follow.

Wide Column Load Method

The characteristics of an orthotropic shell wall are defined by the constitutive relations

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ M_1 \\ M_2 \\ M_T \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & C_{14} & C_{15} & 0 \\ C_{21} & C_{22} & 0 & C_{24} & C_{25} & 0 \\ 0 & 0 & C_{33} & 0 & 0 & C_{36} \\ C_{41} & C_{42} & 0 & C_{44} & C_{45} & 0 \\ C_{51} & C_{52} & 0 & C_{54} & C_{55} & 0 \\ 0 & 0 & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_{12} \\ \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{Bmatrix} \quad \text{(A1)}$$

Equations for the stiffness coefficients C_{ij} are given for a number of different types of shell wall design in Ref. 18. Let

$$\bar{\lambda} = \pi m/(LR)$$
$$p_c = \psi_{mn} E_c R/(1 + \nu_C) \quad \text{(A2)}$$

where ψ_{mn} represents the foundation modulus provided by a solid elastic core. The core modulus E_c is given in Ref. 19. With

$$u_{mn} = [n^2\bar{\lambda}(C_{12} + C_{33})(C_{22} - C_{15}\bar{\lambda}^2 - C_{25}n^2) - \bar{\lambda}(C_{22}n^2 + C_{33}\bar{\lambda}^2)(C_{12} - C_{15}n^2 - C_{14}\bar{\lambda}^2)]/ [(C_{11}\bar{\lambda}^2 + C_{33}n^2)(C_{22}n^2 + C_{33}\bar{\lambda}^2) - n^2\bar{\lambda}^2(C_{12} + C_{33})^2] \quad \text{(A3)}$$

$$v_{mn} = [n\bar{\lambda}^2(C_{12} + C_{33})(C_{12} - C_{15}n^2 - C_{14}\bar{\lambda}^2) - n(C_{11}\bar{\lambda}^2 + C_{33}n^2)(C_{22} - C_{15}\bar{\lambda}^2 - C_{25}n^2)]/ [(C_{11}\bar{\lambda}^2 + C_{33}n^2)(C_{22}n^2 + C_{33}\bar{\lambda}^2) - n^2\bar{\lambda}^2(C_{12} + C_{33})^2] \quad \text{(A4)}$$

The classical buckling load N_{CL} is

$$N_{CL} = [C_{44}\bar{\lambda}^4 + (C_{66} + 2C_{45})\bar{\lambda}^2n^2 + C_{55}n^4 - 2C_{15}\bar{\lambda}^2 - 2C_{25}n^2 + C_{22} + u_{mn}\bar{\lambda}(C_{12} - C_{14}\bar{\lambda}^2 - C_{15}n^2) + v_{mn}n(C_{22} - C_{15}\bar{\lambda}^2 - C_{25}n^2) + pn^2 + p_c]/(n^2 + \bar{\lambda}^2) \quad \text{(A5)}$$

The effective radius to thickness ratio in the computer program is defined as

$$(R/t)_e = [5.46(\bar{C}_{44} + \bar{C}_{55})C_{22}/(C_{11}C_{22} - C_{12}^2)]^{-1/2}$$

with

$$\bar{C}_{44} = C_{44} - (C_{14}^2/C_{11}),\ \bar{C}_{55} = C_{55} - (C_{25}^2/C_{22}) \quad \text{(A6)}$$

If there is no elastic core, the lower bound, or the wide column load, corresponds to buckling in one axial half wave. Hence

$$N_{WC} = R^2\bar{C}_{44}(\pi/L)^2 \quad \text{(A7)}$$

For the infinite shell with an elastic core the lower bound is found (after minimization with respect to the wavelength) to be,

$$N_{WC} = 1.19(R^2\bar{C}_{44})^{1/3}[E_c/(1 - \nu_c^2)]^{2/3} \quad \text{(A8)}$$

The critical buckling load is

$$N_{CR} = N_{WC} + \phi(N_{CL} - N_{WC}) \quad \text{(A9)}$$

where $\phi = f(R/t)_e$, see, for example, Fig. 1.

Symmetric Imperfection Method

The equations for the orthotropic cylinder with an elastic core can easily be derived in a manner following for instance the developments in Ref. 14. The derivation is thus not presented here but only the final equations. It should be noticed first that there exists two different core modulii, one corresponding to the prebuckling displacements and one to the displacement increments connected with buckling. It is assumed in the analysis that the axial wavelength of the buckling pattern is twice the wavelength of the initial imperfection pattern. Hence the two core modulii are $\psi_0 = \psi(2m,0)$ and $\psi_1 = \psi(m,n)$. It is assumed also that the imperfection amplitude for the equivalent monocoque cylinder is applicable to any sinusoidal pattern of imperfections whose wavelength is equal to or larger than the critical wavelength for axisymmetric buckling. The computer program computes first this wavelength (in the presence of an elastic core through iteration). The critical load is then determined for a series of wavelengths until a minimum is found. The half wavelength of the buckling pattern is not allowed to exceed the shell length. In addition, of course, the critical load is minimized with respect to the number of circumferential waves. For a given combination of axial and circumferential wave numbers the critical load can be obtained from the following.

$$\lambda = m\pi/L,\ \bar{C}_{11} = C_{11}C_{22} - C_{12}^2 + \psi_0 R C_{11} \qquad \bar{C}_{14} = 2C_{12}C_{14} - C_{11}C_{24} - C_{11}C_{15} \tag{A10}$$

$$a = C_{11}\bar{C}_{44}\lambda^4 + \lambda^2\bar{C}_{14}/R + \bar{C}_{11}/R^2,\ b = C_{11}\lambda^2 \qquad c = [\bar{C}_{11} - (C_{24}C_{11} - C_{12}C_{14})\lambda^2 R - \phi_1 C_{11} R]/R \tag{A11}$$

$$\begin{aligned}
K_1 &= A_{23}(\lambda/2)^4 + (A_{13} + A_{24})(\lambda/2)^2(n/R)^2 + A_{14}(n/R)^4 \\
K_2 &= A_{11}(n/R)^4 + (2A_{12} + B_{33})(\lambda/2)^2(n/R)^2 + A_{22}(\lambda/2)^4 \\
K_3 &= A_{11}(n/R)^4 + 9(2A_{12} + B_{33})(\lambda/2)^2(n/R)^2 + 81A_{22}(\lambda/2)^4 \\
K_4 &= A_{33}(\lambda/2)^4 + (2A_{34} + B_{66})(\lambda/2)^2(n/R)^2 + A_{44}(n/R)^4
\end{aligned} \tag{A12}$$

Where the A_{ij} are the coefficients in the semi-inverted form of the constitutive equations. For orthotropic shells these are

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_{12} \\ M_1 \\ M_2 \\ M_T \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{13} & A_{14} & 0 \\ A_{12} & A_{22} & 0 & A_{23} & A_{24} & 0 \\ 0 & 0 & B_{33} & 0 & 0 & B_{36} \\ -A_{13} & -A_{23} & 0 & A_{33} & A_{34} & 0 \\ -A_{14} & -A_{24} & 0 & A_{34} & A_{44} & 0 \\ 0 & 0 & -B_{36} & 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \\ \kappa_1 \\ \kappa_2 \\ -\kappa_{12} \end{Bmatrix} \tag{A13}$$

In addition

$$K_5 = \mu\bar{t}\left[\frac{2(\lambda/2)^2(n/R)^2\mu\bar{t} - (1/R)(\lambda/2)^2 - K_1}{K_2} + \frac{2(\lambda/2)^2(n/R)^2\mu\bar{t}}{K_3}\right] \tag{A14a}$$

$$K_6 = \mu\bar{t}C_{11}\lambda^2\left[\mu\bar{t}\,\frac{2(\lambda/2)^2(n/R)^2}{K_2} + \frac{2(\lambda/2)^2(n/R)^2\mu\bar{t} - (1/R)(\lambda/2)^2 - K_1}{K_2} + \frac{4(\lambda/2)^2(n/R)^2\mu\bar{t}}{K_3}\right] \tag{A14b}$$

$$K_7 = 2(\mu\bar{t}\lambda^2 C_{11})^2(\lambda/2)^2(n/R)^2(K_2^{-1} + K_3^{-1}) \tag{A14c}$$

$$B_0 = [K_1 + (1/R)(\lambda/2)^2]\{[2(\lambda/2)^2(n/R)^2\mu\bar{t} - (1/R)(\lambda/2)^2 - K_1]/-K_2\} + K_4 + (\psi_1/R) + pR(n/R)^2 + 2(\lambda/2)^2(n/R)^2K_5 - \psi_0(C_{11}/\bar{C}_{11})p(n/R)^2R \tag{A15a}$$

$$B_1 = 2(\lambda/2)^2(n/R)^2K_6 - 2(\lambda/2)^2(n/R)^2\mu\bar{t}C - 2[K_1 + (1/R)(\lambda/2)^2]C_{11}\lambda^2(\lambda/2)^2(n/R)^2\mu\bar{t}/K_2 \tag{A15b}$$

$$B_2 = (\lambda/2)^2 + \psi_1 R(C_{12}/\bar{C}_{11})(n/R)^2 \tag{A15c}$$

$$B_3 = 2K_7(\lambda/2)^2(n/R)^2 \tag{A15d}$$

$$C = \bar{C}_{11}/R - (C_{24}C_{11} - C_{12}C_{14})\lambda^2 - \psi_0 C_{11} \tag{A16}$$

$$\begin{aligned}
\alpha &= -[(b^2B_0 - bB_1 + 2abB_2 + B_3)/b^2B_2] \\
\beta &= [(2abB_0 - aB_1 + a^2B_2)/b^2B_2] \\
\gamma &= -(a^2B_0/b^2B_2)
\end{aligned} \tag{A17}$$

The critical value of the axial load N_{CR} can now be found as the lowest real root to the equation,

$$N_{CR}^3 + \alpha N_{CR}^2 + \beta N_{CR} + \gamma = 0 \tag{A18}$$

References

[1] Almroth, B. O., Bushnell, D., and Sobel, L. H., "Buckling of Shells of Revolution with Various Wall Constructions, Vol. 1—Numerical Results," CR-1049, May 1968, NASA.

[2] Milligan, R. et al., "General Instability of Orthotropically Stiffened Cylinders," AFFDL TR 65-161, July 1965, Wright-Patterson Air Force Base.

[3] Katz, L., "Compression Tests on Integrally Stiffened Cylinders," TM X-53315, Aug. 1965, NASA.

[4] von Karman, T. and Tsien, H. S., "The Buckling of Thin Cylindrical Shells under Axial Compression," *Journal of the Aeronautical Sciences*, Vol. 8, June 1941, pp. 303–312.

[5] Hoff, N. J., Madsen, W. A., and Mayers, J., "Postbuckling Equilibrium of Axially Compressed Circular Cylindrical Shells," *AIAA Journal*, Vol. 4, No. 1, Jan. 1966, pp. 126–133.

[6] Donnell, L. H. and Wan, C. C., "Effect of Imperfections on Buckling of Thin Cylinders and Columns Under Axial Compression," *Journal of Applied Mechanics*, Vol. 17, March 1950, p. 73.

[7] Harris, L. A. et al., "The Stability of Thin-Walled Unstiffened Circular Cylinders under Axial Compression Including the Effects of Internal Pressure," *Journal of the Aeronautical Sciences*, Vol. 24, Aug. 1957, p. 587.

[8] Burns, A. B. and Almroth, B. O., "Structural Optimization of Axially Compressed, Ring-Stringer Stiffened Cylinders," *Journal of Spacecraft and Rockets*, Vol. 3, No. 1, Jan. 1966, pp. 19–25.

[9] Shanley, F. R., "Simplified Analysis of General Instability of Stiffened Shells in Pure Bending," *Journal of the Aeronautical Sciences*, Vol. 16, Oct. 1949, pp. 590–592.

[10] Peterson, J. P. and Dow, M. B., "Compression Tests on Circular Cylinders Stiffened Longitudinally by Closely Spaced Z-Section Stringers," Memo 2-12-59L, March 1959, NASA.

[11] Smith, G. W. and Spier, E. E., "Study of Stability of Unpressurized Shell Structures Under Static Loading," Rept. GDC DDG 67-005, June 20, 1967, General Dynamics, San Diego, Calif.

[12] Koiter, W. T., *Over de Stabiliteit van het elastisch Evenwicht*, H. J. Paris, Amsterdam, Holland, 1945; translation available as AFFDL-TR-70-25, Feb 1970, Wright-Patterson Air Force Base.

[13] Brush, D. O., "Imperfection Sensitivity of Stringer Stiffened Cylinders," *AIAA Journal*, Vol. 6, No. 12, Dec. 1968, pp. 2445–2447.

[14] Koiter, W. T., "The Effect of Axisymmetric Imperfections on the Buckling of Cylindrical Shells Under Axial Compression," Rept. 6-90-63-86, Aug. 1963, Lockheed Missiles & Space Co.

[15] Almroth, B. O., Burns, A. B., and Pittner, E. V., "Design Criteria for Axially Loaded Cylindrical Shells," Rept. 6-78-69-22, April 1969, Lockheed Missiles & Space Co.

[16] Card, M. F. and Jones, R. M., "Experimental and Theoretical Results for Buckling of Eccentrically Stiffened Cylinders," TN D-3639, Oct. 1966, NASA.

[17] Stuhlman, C., DeLuzio, A., and Almroth, B. O., "Influence of Stiffener Eccentricity and End Moment on Stability of Cylinders in Compression," *AIAA Journal*, No. 5, Vol. 4, May 1966, pp. 872–877.

[18] Bushnell, D., Almroth, B. O., and Sobel, L. H., "Buckling of Shells of Revolution with Various Wall Constructions, Vol. 2—Basic Equations and Method of Solution," CR-1050, May 1968, NASA.

[19] Seide, P., "The Stability Under Axial Compression and Lateral Pressure of Circular Cylindrical Shells with a Soft Elastic Core," *Journal of the Aerospace Sciences*, Vol. 29, July 1962, pp. 851–862.

[20] Kachman, D. R., "Test Report on Buckling of Propellant Cylinders Under Compressive Loads," GM-59-7520.6-24, Nov. 1959, Space Technology Lab.

[21] Fitzgibbon, D. P., "Preliminary Results of Sub-Scale Tests on Cylinders Filled with an Elastic Core," GM-69-7520.6-11, April 1960, Space Technology Lab.

[22] Goree, W. S. and Nash, A. N., "Elastic Stability of Circular Cylindrical Shells Stabilized by a Soft Elastic Core," Interim 4 for Office of Ordnance Research, May 1960, Dept. of Engineering Mechanics, Univ. of Florida.

[23] March, H. W. and Kuenzi, E. W., "Buckling of Cylinders of Sandwich Construction in Axial Compression," Rept. 1830, Dec. 1957, Forest Products Lab., Madison, Wis.

[24] Norris, C. B. and Kuenzi, E. W., "Buckling of Long, Thin, Plywood Cylinders in Axial Compression," Rept. 1322B, Forest Products Lab., Nov. 1943, Madison, Wis.

[25] Tasi, J., Feldman, A., and Stang, D. A., "The Buckling Strength of Filament-Wound Cylinders Under Axial Compression," CR-266, July 1965, NASA.

[26] Holston, A., Jr., Feldman, A., and Stang, D. A., "Stability of Filament Wound Cylinders Under Combined Loading," AFFDL TR 67-55, July 1965, Wright-Patterson Air Force Base.

[27] Card, M. F., "Experiments to Determine the Strength of Filament-Wound Cylinders Loaded in Axial Compression," TN D-3522, Aug. 1966, NASA.

[28] Singer, J., private correspondence, Nov. 1968 Haifa, Israel.

[29] Singer, J., "The Influence of Stiffener Geometry and Spacing on the Buckling of Axially Compressed Cylindrical and Conical Shells," Technion-Israel Institute of Technology, Dept. of Engineering Report TAE-68, Oct. 1967, Air Force Office of Scientific Research.

[30] Becker, H., Gerard, G., and Winter, R., "Experiments on Axial Compressive General Instability of Monolithic Ring-Stiffened Cylinders," *AIAA Journal*, Vol. 1, No. 7, July 1963, pp. 1614–1618.

[31] Lundquist, E. E., "Strength Test of Thin-Walled Duralumin Cylinders in Compression," Rept. 473, 1933, NACA.

[32] Ballerstedt, W. and Wagner, H., "Versuche über die Festigkeit dünner universteifter Zylinder unter Schub-und Längs Kräften," Luftfahrt-Forschung, Vol. 13, Sept. 1936, pp. 309–312.

[33] Kanemitsu, S. and Nojima, N. M., "Axial Compression Test of Thin-Walled Circular Cylinders," thesis submitted for the degree of Master of Science in Aeronautical Engineering to the California Inst. of Technology, 1939.

[34] Weingarten, V. I., Morgan, E. J., and Seide, P., "Elastic Stability of Thin-Walled Cylindrical and Conical Shells Under Combined Internal Pressure and Axial Compression," *AIAA Journal*, Vol. 3, No. 6, June 1965, pp. 1118–1125.

[35] Dow, M. B. and Peterson, J. P., "Bending and Compression Tests of Pressurized Ring-Stiffened Cylinders," TN D-360, April 1960, NASA.

[36] Fung, Y. C., and Sechler, E. E., "Buckling of Thin-Walled Circular Cylinders Under Axial Compression and Internal Pressure," *Journal of the Aeronautical Sciences*, Vol. 24, May 1957, pp. 351–356.

[37] Lo, H., Crate, H., and Schwartz, E. B., "Buckling of Thin Walled Cylinder Under Axial Compression and Internal Pressure," Rept. 1027, 1951, NACA.

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Adjustment of a Thermal Mathematical Model to Test Data

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An iteration method is presented by which a thermal mathematical model of a spacecraft can be adjusted to measured temperatures by minimizing the sum of the squares of the residuals of the nodal heat balance equations. The method permits a common treatment of three cases, in which the number of unknowns in the system of equations is smaller than, equal to, or larger than the number of equations. These cases may occur in any of four conditions of test; 1) steady-state, 2) cooling from steady-state, 3) cyclic state (repeated, periodic, cooling and heating), and 4) arbitrary transient heating. The 93-node model of the ESRO-I satellite is analyzed. The foregoing four test conditions and a total number of 16,089 temperatures at two attitudes of the spacecraft (horizontal and vertical position) with respect to solar radiation are used to recalculate 7 sets of 1957 unknown factors with a digital computer. The adjustment reduces the sum of the residuals of the heat balance equations in the steady-state tests by a factor of 5, and in all transient tests by a factor of 2. The rms of the differences between the measured and calculated temperatures and the standard deviation of these differences are also reduced by a factor of 2 when a model found from a cyclic state is applied to other tests. It is observed that the best verification of any spacecraft model is possible from such a state.

Nomenclature

a = known coefficient in general system of equations
A = radiation factor to space, w/(°K)^4
A' = area, m^2
b = known factor in general system of equations
B = heat capacity, w-sec/°K
B' = absorption or "Gebhart" factor
c = specific heat, w-sec/kg-°K
C = thermal conductance, w/°K
D = temperature-time derivative, °K/sec
E = "Earth" radiation, w
FE,FS = weighting factors defining how the corrections are distributed among the unknowns E, S, respectively
G = matrix used in Eqs. (4) and (5)
IAV = integer array specifying how the equations are to be averaged
KOI = integer specifying the type of test
KTE = number of equations for which $S = 0$ in the cyclic state
M = number of instants or moments
N = number of nodes
NAV = number of averaged equations
NC = number of independent conductances
NE = number of equations in the general system
NPO = number of points on a temperature-time curve
NU = number of unknowns in the general system
NR = number of independent radiation factors
NS = number of skin nodes
r = residual in heat balance equation, w
R = radiation factor, w/(°K)^4
RE,RS = ranges specifying the hypervolume in which the unknowns E, S, respectively, are allowed to vary
s = sum of all terms in the heat balance equation, w
S,S' = solar radiation, w, and solar flux, w/m^2, respectively
t = time, sec

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